An improved Gauge Unfixing formalism and the Abelian Pure Chern Simons Theory

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Abstract

We propose a variant scheme of the Gauge Unfixing formalism which modifies directly the original phase space variables of a constrained system. These new variables are gauge invariant quantities. We apply our procedure in a mixed constrained system that is the Abelian Pure Chern Simons Theory where several gains are obtained. In particular, from the gauge invariant Hamiltonian and using the inverse Legendre transformation, we obtain the same initial Abelian Pure Chern Simons Lagrangian as the gauge invariant Lagrangian. This result shows that the gauge symmetry of the action is certainly preserved.

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1 Introduction

The Abelian Pure Chern Simons (CS) Theory is a mixed constrained system where one of their four constraints must be redefined in order to be a first class one. Then, after this step, we have well defined algebras of two first class constraints and two second class constraints. The BFT formalism[1, 2], which enlarges the phase space variables with the introduction of the Wess Zumino (WZ) fields, has been used with the objective to embed the CS theory[3]. As a result, the authors show many important features. Another work[4] has also employed the BFT formalism to study a Nonabelian version of the CS theory. In this article, the authors propose two methods that overcome the problem of embedding mixed constrained systems. In an opposite side of the BFT formalism, there is another method that embeds second class constrained systems, called Gauge Unfixing (GU) formalism. It was proposed by Mitra and Rajaraman[5] and continued by Vytheeswaran[6, 7]. This formalism considers part of the total second class constraints as the gauge symmetry generators while the remaining ones form the gauge fixing terms. The second class Hamiltonian must be modified in order to satisfy a first class algebra with the constraints initially chosen to be the gauge symmetry generators. This approach has an elegant property that does not extend the phase space with extra variables.

The purpose of this paper is to give a alternative scheme for the GU formalism and to apply this method to the CS theory. Our aim is to redefine the original phase space variables of a constrained system, without to introduce any WZ terms, in order to be gauge invariant fields. Then, after this procedure, we will construct functions of these gauge invariant fields which will be gauge invariant quantities. As we will see, we begin with a mixed constrained system that is the CS theory and, applying our formalism, we obtain a first class system written only in terms of the original phase space variables with many novel features. As many important constrained systems have only two second class constraints, then, in principle, we present our formalism only for systems with two second class constrains without any loss of generality. In order to clarify the exposition of the subject, this paper is organized as follows: in Section 2, we give a short review of the usual GU formalism. In Section 3, we present our formalism. In Section 4, we apply our procedure to the CS theory. In Section 5, we make our concluding remarks.

2 A brief review of the Gauge Unfixing formalism

Let us consider a constrained system described by the second class Hamiltonian H and two second class constraints T_1 and T_2. The basic idea of the GU formalism is to select one of the two second class constraints to be the gauge symmetry generator. As example, if we choose T_1 as the first class constraint, then, we need to scale T_1 as $\frac{T_1}{\Delta_{12}} \equiv \tilde{T}$ where $\Delta_{12} = \{T_1, T_2\}$. The second class constraint T_2 will be discarded. The Poisson bracket between $\tilde{T}$ and T_2 is $\{\tilde{T}, T_2\} = 1$, so that $\tilde{T}$ and T_2 are canonically conjugate. The second class Hamiltonian must be modified in order to satisfy a first class algebra. Then, the gauge invariant Hamiltonian is building by the series in powers of T_2

$$\tilde{H} = H + T_2\,\{H, \tilde{T}\} + \frac{1}{2!}T_2^2\,\{\{H, \tilde{T}\}, \tilde{T}\} + \frac{1}{3!}T_2^3\,\{\{\{H, \tilde{T}\}, \tilde{T}\}, \tilde{T}\} + \ldots, \quad (1)$$

where we can show that $\{\tilde{H}, \tilde{T}\} = 0$ and $\tilde{T}$ must satisfy a first class algebra $\{\tilde{T}, \tilde{T}\} = 0$. The gauge invariant Hamiltonian, Eq.(1), can be elegantly written in terms of a projection operator on the second class Hamiltonian H

$$\tilde{H} = e^{T_2 \tilde{T}_{op}} : H, \quad (2)$$

where $\tilde{T}_{op}\, H \equiv \{H, \tilde{T}\}$ and an ordering prescription must be adopted that is T_2 must come before the Poisson bracket.

3 The improved Gauge Unfixing formalism

Let us start with the original phase space variables written as

$$F = (q_i, p_i), \quad (3)$$

where F can describe a particle or field model. As we haven seen in Section 2, the usual GU formalism embeds directly the second class Hamiltonian. Thus, our strategy is to construct a gauge invariant function $\tilde{A}$ from the second class function A by gauging the original phase space variables, using for this the idea of the GU formalism.

Denoting the first class variables by

$$\tilde{F} = (\tilde{q}_i, \tilde{p}_i), \quad (4)$$

we determine the first class function $\tilde{F}$ in terms of the original phase space variables by employing the variational condition

$$\delta\tilde{F} = \epsilon\{\tilde{F}, \tilde{T}\} = 0, \quad (5)$$

where $\tilde{T}$ is the scaled second class constraint chosen to be the gauge symmetry generator and ϵ is an infinitesimal parameter. Any function of $\tilde{F}$ will be gauge invariant since

$$\{\tilde{A}(\tilde{F}), \tilde{T}\} = \{\tilde{F}, \tilde{T}\}\frac{\partial\tilde{A}}{\partial\tilde{F}} = 0, \quad (6)$$

where

$$\{\tilde{F}, \tilde{T}\}\frac{\partial\tilde{A}}{\partial\tilde{F}} \equiv \{\tilde{q}_i, \tilde{T}\}\frac{\partial\tilde{A}}{\partial\tilde{q}_i} + \{\tilde{p}_i, \tilde{T}\}\frac{\partial\tilde{A}}{\partial\tilde{p}_i}. \quad (7)$$

Consequently, we can obtain a gauge invariant function from the replacement of

$$A(F) \Rightarrow A(\tilde{F}) = \tilde{A}(\tilde{F}). \tag{8}$$

The gauge invariant phase space variables $\tilde{F}$ are constructed by the series in powers of T_2

$$\tilde{F} = F + \sum_{n=1}^{\infty} c_n\, T_2^n = F + c_1\, T_2 + c_2\, T_2^2 + \ldots, \tag{9}$$

where this series has an important boundary condition that is

$$\tilde{F}(T_2 = 0) = F. \tag{10}$$

The condition above and the relation (8) show that when we impose the discarded constraint T_2 equal to zero, we reobtain the original second class system. Therefore, the relations (8) and (10) guarantee the equivalence between our first class model and the initial second class system.

The coefficients c_n in the relation (9) are then determined by the variational condition, Eq.(5). The general equation for c_n is

$$\delta\tilde{F} = \delta F + \sum_{n=1}^{\infty} (\delta c_n\, T_2^n + n\, c_n\, T_2^{(n-1)} \delta T_2) = 0, \tag{11}$$

where

$$\begin{aligned} \delta F &= \epsilon\{F, \tilde{T}\}, & (12)\\ \delta c_n &= \epsilon\{c_n, \tilde{T}\}, & (13)\\ \delta T_2 &= \epsilon\{T_2, \tilde{T}\} = -\,\epsilon. & (14) \end{aligned}$$

In Eq.(14) we assume that $\{\tilde{T}, T_2\} = 1$. Then, for the linear correction term $(n = 1)$, we have

$$\delta F + c_1\, \delta T_2 = 0 \;\Rightarrow\; \delta F - c_1\, \epsilon = 0 \;\Rightarrow\; c_1 = \frac{\delta F}{\epsilon}. \tag{15}$$

For the quadratic correction term (n=2), we get

$$\delta c_1 + 2c_2\, \delta T_2 = 0 \;\Rightarrow\; \delta c_1 - 2c_2 \epsilon = 0 \;\Rightarrow\; c_2 = \frac{1}{2} \frac{\delta c_1}{\epsilon}. \tag{16}$$

For $n \geq 2$, the general relation is

$$\begin{aligned} &\delta c_n + (n+1) c_{n+1}\, \delta T_2 = 0 \;\Rightarrow\; \delta c_n - (n+1)\, c_{(n+1)} \epsilon = 0 \\ &\qquad\qquad\Rightarrow\; c_{(n+1)} = \frac{1}{(n+1)} \frac{\delta c_n}{\epsilon}. \end{aligned} \tag{17}$$

Using the relations (15), (16) and (17) we obtain the series which determines $\tilde{F}$

$$\tilde{F} = F + T_2 \frac{\delta F}{\epsilon} + \frac{1}{2!} T_2^2 \frac{\delta\delta F}{\epsilon^2} + \frac{1}{3!} T_2^3 \frac{\delta\delta\delta F}{\epsilon^3} + \ldots . \tag{18}$$

The expression $\tilde{F}$ can also be elegantly written in terms of a projection operator on F

$$\tilde{F} = e^{T_2 \frac{\delta}{\epsilon}} : F, \tag{19}$$

where again an ordering prescription must be adopted that is T_2 must come before $\frac{\delta}{\epsilon}$. Now, if we calculate the Poisson bracket between the two gauge invariant variables defined by the formula (18) and next we take the limit $T_2 \rightarrow 0$, we get

$$\begin{aligned} \{\tilde{F}, \tilde{G}\}_{T_2=0} &= \{F, G\} + \{F, \tilde{T}\}\{T_2, G\} - \{F, T_2\}\{\tilde{T}, G\} \\ &= \{F, G\} + \{F, T_i\}\epsilon^{ij}\{T_j, G\}. \end{aligned} \tag{20}$$

If we assume that $\{T_i, T_j\} \equiv \Delta_{ij} = \epsilon_{ij}$, being $T_1 \equiv \tilde{T}$, we can write

$$\begin{aligned} \{\tilde{F}, \tilde{G}\}_{T_2=0} &= \{F, G\} + \{F, T_i\}\Delta^{ij}\{T_j, G\} \\ &= \{F, G\}_D, \end{aligned} \tag{21}$$

where $\Delta^{ij} \equiv \epsilon^{ij}$ is the inverse of $\Delta_{ij} \equiv \epsilon_{ij}$ and $\{F, G\}_D$ is the Dirac bracket[8]. Thus, we can observe that when we change the gauge invariant variables by imposing the condition $T_2 = 0$, we return to the original second class system where the Poisson brackets transform in the Dirac brackets. This important result also confirms the consistency of our formalism. The same result was obtained by employing the BFT formalism[3].

4 The Abelian Pure Chern Simons Theory

The CS theory, being a three dimensional field theory, is governed by the Lagrangian

$$L = \int d^2x \, \frac{k}{2} \, \epsilon_{\mu\nu\rho} \, A^\mu \partial^\nu A^\rho, \tag{22}$$

where k is a constant. From the standard Dirac constrained formalism[8] we obtain three canonical momenta which are the primary constraints

$$T_0 \equiv \pi_0 \approx 0, \tag{23}$$

$$T_i \equiv \pi_i - \frac{k}{2} \, \epsilon_{ij} A^j \approx 0. \;\; (i = 1, 2) \tag{24}$$

Using the Legendre transformation we derive the canonical Hamiltonian

$$H_c = -k \int d^2x\, A^0\, \epsilon_{ij}\, \partial^i A^j. \tag{25}$$

From the temporal stability condition of the constraint, Eq.(23), we get the secondary constraint

$$T_3 \equiv k\, \epsilon_{ij}\, \partial^i A^j \approx 0. \tag{26}$$

We observe that no further constraints are generated via this iterative procedure. T_0, T_i and T_3 are the total constraints of the model . In order to separate the second and the first class constraints, we need to redefine the constraint (26), in principle, by using a trial and error procedure

$$\tilde{T}_3 \equiv T_3 + \partial^i T_i = \partial^i \pi_i + \frac{k}{2}\, \epsilon_{ij}\, \partial^i A^j. \tag{27}$$

Then, T_0 and $\tilde{T}_3$ form the first class constraints, while T_i, Eq.(24), forms the second class constraints satisfying the algebra

$$\{T_i(x), T_j(y)\} = -k\, \epsilon_{ij}\, \delta^2(x-y). \;\; (i,j = 1,2) \tag{28}$$

Our formalism begins by choosing the symmetry gauge generator as

$$\tilde{T} = -\frac{T_1}{k} = -\frac{\pi_1}{k} + \frac{A^2}{2}, \tag{29}$$

where we have the algebra $\{\tilde{T}(x), T_2(y)\} = \delta^2(x-y)$. The second class constraint $T_2 = \pi_2 + \frac{k}{2} A^1$ will be discarded. The infinitesimal gauge transformations generated by symmetry generator $\tilde{T}$ are

$$\delta A^i = \epsilon \{A^i(x), \tilde{T}(y)\} = -\frac{\epsilon}{k}\, \delta^i_1\, \delta^2(x-y), \tag{30}$$

$$\delta \pi_i = \epsilon \{\pi_i(x), \tilde{T}(y)\} = -\frac{\epsilon}{2}\, \delta^2_i\, \delta^2(x-y), \tag{31}$$

$$\delta T_2 = \epsilon\, \{T_2(x), \tilde{T}(y)\} = -\epsilon\, \delta^2(x-y). \tag{32}$$

The gauge invariant field $\tilde{A}^i$ is constructed by the series in powers of T_2

$$\tilde{A}^i = A^i + b_1\, T_2 + b_2\, T_2^2 + \ldots + b_n\, T_2^n. \tag{33}$$

From the invariance condition $\delta \tilde{A}^i = 0$, we can compute all the correction terms b_n. For the linear correction term in order of T_2, we get

$$\delta A^i + b_1 \delta T_2 = 0 \;\Rightarrow\; -\frac{\epsilon}{k}\, \delta^i_1 \delta^2(x-y) - b_1 \epsilon\, \delta^2(x-y) = 0 \;\Rightarrow\; b_1 = -\frac{1}{k}\, \delta^i_1. \tag{34}$$

For the quadratic term, we obtain $b_2 = 0$, since $\delta b_1 = \epsilon\{b_1, \tilde{T}\} = 0$. Due to this, all the correction terms b_n with $n \geq 2$ are null. Therefore, the gauge invariant field $\tilde{A}^\mu$ is

$$\begin{aligned}\tilde{A}^0 &= A^0, &(35)\\ \tilde{A}^i &= A^i - \frac{1}{k}\,\delta_1^i\,T_2, &(36)\end{aligned}$$

or

$$\begin{aligned}\tilde{A}^0 &= A^0, &(37)\\ \tilde{A}^1 &= A^1 - \frac{1}{k}\,T_2, &(38)\\ \tilde{A}^2 &= A^2, &(39)\end{aligned}$$

where by using Eq.(30), it is easy to show that $\delta\tilde{A}^\mu = 0$. The gauge invariant field $\tilde{\pi}_i$ is also constructed by the series in powers of T_2

$$\tilde{\pi}_i = \pi_i + c_1\,T_2 + c_2\,T_2^2 + \ldots + c_n\,T_2^n. \tag{40}$$

From the invariance condition $\delta\tilde{\pi}_i = 0$, we can compute all the correction terms c_n. For the linear correction term in order of T_2, we get

$$\delta\pi_i + c_1\delta T_2 = 0 \;\Rightarrow\; -\frac{\epsilon}{2}\delta_i^2\delta^2(x-y) - c_1\epsilon\,\delta^2(x-y) = 0 \;\Rightarrow\; c_1 = -\frac{1}{2}\delta_i^2. \tag{41}$$

For the quadratic term, we obtain $c_2 = 0$, since $\delta c_1 = \epsilon\{c_1, \tilde{T}\} = 0$. Due to this, all the correction terms c_n with $n \geq 2$ are null. Therefore, the gauge invariant field $\tilde{\pi}_i$ is

$$\tilde{\pi}_i = \pi_i - \frac{1}{2}\,\delta_i^2\,T_2, \tag{42}$$

or

$$\begin{aligned}\tilde{\pi}_1 &= \pi_1, &(43)\\ \tilde{\pi}_2 &= \pi_2 - \frac{1}{2}\,T_2, &(44)\end{aligned}$$

where by using Eq.(31), it is easy to show that $\delta\tilde{\pi}_i = 0$. The Poisson brackets between the gauge invariant fields are

$$\begin{aligned}\{\tilde{A}^i(x), \tilde{A}^j(y)\} &= \frac{1}{k}\,\epsilon^{ij}\,\delta^2(x-y), &(45)\\ \{\tilde{\pi}_i(x), \tilde{\pi}_j(y)\} &= \frac{k}{4}\,\epsilon_{ij}\,\delta^2(x-y), &(46)\\ \{\tilde{A}^i(x), \tilde{\pi}_j(y)\} &= \frac{1}{2}\,\delta_j^i\;\delta^2(x-y), &(47)\end{aligned}$$

which are the same, as discussed in Eq.(21), derived by the Dirac brackets[3]. The gauge invariant Hamiltonian, written only in terms of the original phase space variables, is obtained by substituting A^μ by $\tilde{A}^\mu$, Eqs.(35) and (36), in the canonical Hamiltonian, Eq.(25), as follows

$$\begin{aligned} \tilde{H} &= k \int d^2x \;\; \partial^i \tilde{A}^0 \, \epsilon_{ij} \, \tilde{A}^j = H_c + \int d^2x \; \partial^2 A^0 \; T_2 \\ &= \int d^2x \; [k \, \partial^i A^0 \, \epsilon_{ij} \, A^j + \partial^2 A^0 \, \pi_2 + \frac{k}{2} \, \partial^2 A^0 \, A^1]. \end{aligned} \tag{48}$$

Imposing the temporal stability condition of π_0 $(T_0 \equiv \pi_0)$

$$\begin{aligned} \{\pi_0, \tilde{H}\} = 0 \;\Rightarrow\; k \, \epsilon_{ij} \, \partial^i A^j + \partial^2 \pi_2 + \frac{k}{2} \partial^2 A^1 &= k \, \epsilon_{ij} \, \partial^i A^j + \partial^2 T_2 = 0 \\ &\Rightarrow k \, \epsilon_{ij} \, \partial^i \tilde{A}^j = 0, \end{aligned} \tag{49}$$

we get the secondary constraint

$$\tilde{T}_3 \equiv k \, \epsilon_{ij} \, \partial^i \tilde{A}^j, \tag{50}$$

that is just the secondary constraint, Eq.(26), with the replacement of A^i by $\tilde{A}^i$. The gauge invariant Hamiltonian $\tilde{H}$ and the irreducible constraints $T_0, \tilde{T}$ and $\tilde{T}_3$ form a set of first class algebra given by

$$\{\tilde{H}, \tilde{T}\} = 0, \tag{51}$$
$$\{\tilde{H}, T_0\} = \tilde{T}_3, \tag{52}$$
$$\{\tilde{H}, \tilde{T}_3\} = 0, \tag{53}$$
$$\{\tilde{T}, \tilde{T}_3\} = 0, \tag{54}$$
$$\{\tilde{T}, T_0\} = 0, \tag{55}$$
$$\{T_0, \tilde{T}_3\} = 0, \tag{56}$$

where we have used relation (45) to prove Eq.(53) and the condition $\delta \tilde{A}^i = 0$ to prove Eq.(54). Here, we would like to mention important results obtained by our formalism. First, by imposing the temporal stability of T_0, Eq.(49), we get, by a systematic way, i.e. without to use any trial and error procedure, an irreducible first class constraint $\tilde{T}_3$. Second, by using our improved GU formalism, we do not need to reduce all the constraints of the CS theory in a second class nature. We only embed the initial second class constraints T_i, Eq.(24), and, consequently, we have all the constraints forming a first class set.

Finally, the gauge invariant CS Lagrangian can be deduced by performing the inverse Legendre transformation

$$L = \int d^2x \; (\pi_1 \dot{A}^1 + \pi_2 \dot{A}^2 - \tilde{H}), \tag{57}$$

where we have used $\pi_0 = 0$ and $\tilde{H}$ is given by Eq.(48). The constraint $\tilde{T}$, Eq.(29), leads to the relation for the momentum π_1 given by

$$\tilde{T} = -\frac{\pi_1}{k} + \frac{A^2}{2} = 0 \Rightarrow \pi_1 = \frac{k}{2}A^2. \tag{58}$$

The Hamilton equation of motion produces a relation for $\partial^0 A^2$ given by

$$\partial^0 A^2 = \{A^2, \tilde{H}\} = \partial^2 A^0. \tag{59}$$

Then, using the Eqs. (58) and (59) in the first class Lagrangian, Eq.(57), we get

$$\begin{aligned} L &= \int d^2x \, (\frac{k}{2} A^2 \partial^0 A^1 - k \, \partial^i A^0 \epsilon_{ij} A^j - \frac{k}{2} \, A^1 \partial^2 A^0) \\ &= \int d^2x \, (k \, A^0 \epsilon_{ij} \, \partial^i A^j + \frac{k}{2} A^2 \partial^0 A^1 - \frac{k}{2} \, A^1 \partial^0 A^2) \\ &= \int d^2x \, (k \, A^0 \epsilon_{0ij} \, \partial^i A^j + \frac{k}{2} \, \epsilon_{i0j} \, A^i \partial^0 A^j) \\ &= \int d^2x \, \frac{k}{2} \, \epsilon_{\mu\nu\rho} \, A^\mu \partial^\nu A^\rho. \end{aligned} \tag{60}$$

Eq.(60) is identical to the original Lagrangian, Eq.(22). The same result can be derived by substituting A^μ by $\tilde{A}^\mu$, Eqs.(35) and (36), in the original Lagrangian, Eq.(22). The relation (60) is also an important result because without the presence of the extra terms in the gauge invariant Lagrangian, the original gauge symmetry transformation $A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$ is certainly maintained.

5 Conclusions

In this paper, we have improved the GU formalism by gauging the original phase space variables of a constrained system. In the case of a system with two second class constraints, one of the constraints will be chosen to form the scaled gauge symmetry generator while the other will be discarded. The discarded constraint is used to construct a series for the gauge invariant fields. Consequently, any functions of the gauge invariant fields are gauge invariant quantities. We apply our formalism to the CS model where new results are obtained. Our improved GU formalism can also be used to study the Nonabelian version of the Chern Simons theory[4, 9].

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